SOME GROUP ACTIONS ON $K(x_1, x_2, x_3)$

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ABSTRACT

Let K be any field which may not be algebraically closed, $K(x_1, x_2, x_3)$ be the rational function field of three variables over K, and σ: $K(x_1, x_2, x_3) \longrightarrow K(x_1, x_2, x_3)$ be a K-automorphism defined by

$$\sigma \cdot x_i = (a_i x_i + b_i)/(c_i x_i + d_i)$$

where $a_i, b_i, c_i, d_i \in K$ and $a_i d_i - b_i c_i \neq 0$. Let $\sigma_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \in PGL_2(K)$, $f_i(T) = T^2 - (a_i + d_i)T + (a_i d_i - b_i c_i) \in K[T]$ be the "characteristic polynomial" of σ_i.

THEOREM: *Assume that* $\operatorname{char} K \neq 2$. *Then the fixed field* $K(x_1, x_2, x_3)^{<\sigma>}$ *is not rational (=purely transcendental) over* K *if and only if (i) for each* $1 \leq i \leq 3$, $f_i(T)$ *is irreducible; (ii) the Galois group of* $f_1(T)f_2(T)f_3(T)$ *over* K *is of order 8; and (iii) for each* $1 \leq i \leq 3$, $\operatorname{ord}(\sigma_i)$ *is an even integer.*

1. Introduction

Let K be any field, $K(x_1, \ldots, x_n)$ be the rational function field of n variables over K. If G is a finite subgroup of $GL_n(K)$, then G acts naturally on $K(x_1, \ldots, x_n)$ by K-automorphisms defined by $\sigma \cdot x_j = \sum_{1 \leq i \leq n} a_{ij} x_i$ if $\sigma = (a_{ij})_{1 \leq i,j \leq n} \in G$. Noether's problem asks: under what situations is the fixed field $K(x_1, \ldots, x_n)^G = \{f \in K(x_1, \ldots, x_n) : \sigma \cdot f = f \text{ for any } \sigma \in G\}$ rational (=purely transcendental) over K? Noether's problem is related to the existence of the generic Galois G-extensions over K, which will imply an affirmative answer to the inverse Galois problem in case the base field is an algebraic

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number field. For a survey of Noether's problem and related topics, see [Sw2, Sa2, Ke, Ka2].

In solving Noether's problem, it is almost inevitable to consider the rationality problem of group actions other than finite subgroups of $GL_n(K)$. For example, consider the K-automorphism σ: $K(x_1, \ldots, x_n) \longrightarrow K(x_1, \ldots, x_n)$ defined by $\sigma \cdot x_i = a_i/x_i$ where $a_i \in K \setminus \{0\}$. Giles and McQuillan provided the first answer to the rationality problem of this kind.

THEOREM 1.1 (Giles and McQuillan [GQ]): *Let K be any field and σ: $K(x,y) \longrightarrow K(x,y)$ be a K-automorphism defined by $\sigma \cdot x = a/x$, $\sigma \cdot y = b/y$ where $a, b \in K \setminus \{0\}$. Then $K(x,y)^{<\sigma>}$ is rational over K.*

In [GQ] only the rationality is asserted; the explicit generators for $K(x,y)^{<\sigma>}$ are given in

THEOREM 1.2 ([HK1, (2,7) Lemma; Ka1, Theorem 2.4]): *Let the assumptions be the same as in Theorem 1.1. Then $K(x,y)^{<\sigma>} = K(u,v)$ where*

$$u = \frac{x - (a/x)}{xy - (ab/xy)}, \quad v = \frac{y - (b/y)}{xy - (ab/xy)}.$$

The situation of $n = 3$ is rather intricate and is solved by Saltman.

THEOREM 1.3 (Saltman [Sa3]): *Let K be any field with $\operatorname{char} K \neq 2$, and σ: $K(x_1, x_2, x_3) \longrightarrow K(x_1, x_2, x_3)$ be a K-automorphism defined by $\sigma \cdot x_i = a_i/x_i$ where $a_i \in K \setminus \{0\}$. If $[K(\sqrt{a_1}, \sqrt{a_2}, \sqrt{a_3}) : K] = 8$. Then $K(x_1, x_2, x_3)^{<\sigma>}$ is not retract rational over K.*

Note that rationality implies stable rationality, and stable rationality implies retract rationality.

We remark that the article [Sa3] exemplifies elegantly the machinery of retract rationality developed by Saltman in [Sa1] via solving this concrete case that $K(x_1, x_2, x_3)^{<\sigma>}$ is not retract rational; meanwhile, a necessary and sufficient condition for $K(x_1, \ldots, x_n)^{<\sigma>}$ to be rational (if $n \geq 3$) can be deduced from Theorem 1.3 easily. (See Theorem 4.4.) On the other hand, in view of Theorem 1.3, it will be interesting to consider the general case:

$$\sigma \cdot x_i = (a_i x_i + b_i)/(c_i x_i + d_i),$$

i.e. the K-automorphism provided by the projective transformation on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Note that the answer to the 2-dimensional case is always affirmative. (See Theorem 2.3.) The main results of this paper are the following.

THEOREM 1.4: *Let K be any field with char $K \neq 2$, and σ: $K(x_1, x_2, x_3) \longrightarrow K(x_1, x_2, x_3)$ be a K-automorphism defined by $\sigma \cdot x_i = (a_i x_i + b_i)/(c_i x_i + d_i)$ where $a_i, b_i, c_i, d_i \in K$ and $a_i d_i - b_i c_i \neq 0$. Let*

$$\sigma_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \in PGL_2(K),$$

and $f_i(T) = T^2 - (a_i + d_i)T + (a_i d_i - b_i c_i) \in K[T]$ be the "characteristic polynomial" of σ_i. Then $K(x_1, x_2, x_3)^{<\sigma>}$ is not rational over K if and only if (i) *for each $1 \le i \le 3$, $f_i(T)$ is irreducible;* (ii) *the Galois group of $f_1(T)f_2(T)f_3(T)$ over K is of order 8; and* (iii) *for each $1 \le i \le 3$, $\mathrm{ord}(\sigma_i)$ is an even integer. If $K(x_1, x_2, x_3)^{<\sigma>}$ is not rational, it is not retract rational over K.*

THEOREM 1.5: *Let the assumptions be the same as in Theorem 1.4 except that we now assume* char $K = 2$. *Then $K(x_1, x_2, x_3)^{<\sigma>}$ is always rational over K except possibly the case where, for each $1 \le i \le 3$, $f_i(T)$ is irreducible and is of the form $T^2 - a_i$.*

Note that, in the exceptional case of the above Theorem 1.5, we can find $y_1, y_2, y_3 \in K(x_1, x_2, x_3)$ such that $K(x_1, x_2, x_3) = K(y_1, y_2, y_3)$ and $\sigma \cdot y_i = a_i/y_i$. We do not know the answer to this exceptional case.

We explain briefly the idea of the proof of Theorem 1.4. We embed $K(x_1, x_2, x_3)$ in $K(v_1, w_1, v_2, w_2, v_3, w_3)$ by $x_i = v_i/w_i$. We extend the action by $\sigma(v_i) = a_i v_i + b_i w_i, \sigma(w_i) = c_i v_i + d_i w_i$. The case when $f_i(T)$ is reducible or when $\mathrm{ord}(\sigma_i)$ is infinite for some i is easy and is treated first. So we may consider only the situation when all $f_i(T)$ are irreducible. Let L be the splitting field of $f_1(T)f_2(T)f_3(T)$ over K. While the action of σ on $K(x_i)$ is fractional linear, the action becomes linear in $L(x_i)$, i.e. there exist X_1, X_2, X_3 such that $L(x_1, x_2, x_3) = L(X_1, X_2, X_3)$ and $\sigma(X_i) = \lambda_i X_i$ where $\lambda_i \in L$ and $\mathrm{ord}(\lambda_i) = \mathrm{ord}(\sigma_i)$. Carrying out a long and tedious analysis of these orders, we are able to find $z_1, z_2, z_3 \in < X_1, X_2, X_3 >$ such that $L(X_1, X_2, X_3)^{<\sigma>} = L(z_1, z_2, z_3)$. Since $K(x_1, x_2, x_3)^{<\sigma>} = L(z_1, z_2, z_3)^G$ where $G = \mathrm{Gal}(L/K)$, the question is reduced to the birational classification of algebraic tori. Apply Voskresenskii's Theorem on two-dimensional algebraic tori and Kunyavskii's Theorem on three-dimensional algebraic tori. We will get a rationality criterion. The above method is valid not only for fields K with char $K \neq 2$, but also for the characteristic two case provided that all the polynomials $f_i(T)$ are separable. We then use Saltman's Theorem combined with Kunyavskii's Theorem to show that, if some fixed field is not rational, then it is not retract rational.

STANDING NOTATION. In this paper, K will always denote a field; it is unnecessary to assume that char $K = 0$ or K is algebraically closed. $K(x_1, \ldots, x_n)$ denotes the rational function field of n variables over K. A field extension L of K is rational over K if L is purely transcendental over K, i.e. $L \simeq K(x_1, \ldots, x_n)$ for some n; L is called stably rational over K if there exist elements $y_1, \ldots, y_N$, algebraically independent over L such that $L(y_1, \ldots, y_N)$ is rational over K. The reader is referred to [Sa1] for the definition of retract rationality. If g is an element in some group G, $\mathrm{ord}(g)$ denotes the order of g; in particular, if λ is some non-zero element in a field, $\mathrm{ord}(\lambda) < \infty$ is nothing but that λ is a root of unity.

2. Generalities

First we recall several results which will be used repeatedly throughout this article.

THEOREM 2.1 ([HK2, Theorem 1]): *Let G be a finite group acting on $L(x_1, \ldots, x_n)$, the rational function field of n variables over a field L. Suppose that*

(i) *for any $\sigma \in G, \sigma(L) \subset L$;*
(ii) *the restriction of the actions of G to L is faithful;*
(iii) *for any $\sigma \in G$,*

$$\begin{pmatrix} \sigma(x_1) \\ \vdots \\ \sigma(x_n) \end{pmatrix} = A(\sigma) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + B(\sigma)$$

where $A(\sigma) \in GL_n(L)$ and $B(\sigma)$ is on $n \times 1$ matrix over L.

Then there exist $z_1, \ldots, z_n \in L(x_1, \ldots, x_n)$ such that

$$L(x_1, \ldots, x_n) = L(z_1, \ldots, z_n)$$

with $\sigma(z_i) = z_i$ for any $\sigma \in G$, any $1 \le i \le n$.

THEOREM 2.2 ([AHK, Theorem 3.1]): *Let G be any group whose order may be finite or infinite. Suppose that G acts on $L(x)$, the rational function field of one variable over a field L, such that, for any $\sigma \in G, \sigma(L) \subset L$ and $\sigma(x) = a_\sigma \cdot x + b_\sigma$ for some $a_\sigma, b_\sigma \in L$ with $a_\sigma \neq 0$. Then $L(x)^G = L^G$ or $L^G(f(x))$ where $f(x) \in L[x]$ is of positive degree.*

THEOREM 2.3 ([AHK, Theorem 1.4]): *Let K be any field and $\sigma\colon K(x_1, x_2) \longrightarrow K(x_1, x_2)$ be a K-automorphism defined by $\sigma \cdot x_i = (a_i x_i + b_i)/(c_i x_i + d_i)$ where $a_i, b_i, c_i, d_i \in K$ with $a_i d_i - b_i c_i \neq 0$. Then $K(x_1, x_2)^{<\sigma>}$ is rational over K.*

Finally, we recall two results about algebraic tori. For general notions of algebraic tori we refer to Voskresenskii's monograph [Vo2]. Here, we just give an algebraic formulation of an algebraic torus defined over a field K: Let L be a finite Galois extension of K with Galois group G, $L(x_1, \ldots, x_n)$ be the rational function field of n variables over L, and $\rho\colon G \longrightarrow GL_n(\mathbb{Z})$ be a group homomorphism. Then the action of G on L can be extented to $L(x_1, \ldots, x_n)$ by

$$\tau \cdot x_j = \prod_{1 \le i \le n} x_i^{n_{ij}}$$

where $\rho(\tau) = (n_{ij})_{1 \le i,j \le n} \in GL_n(\mathbb{Z})$ for any $\tau \in G$. The fixed field $L(x_1, \ldots, x_n)^G$ is the function field of some n-dimensional algebraic torus defined over K and split by L.

THEOREM 2.4 (Voskresenskii [Vo1]): *All the two-dimensional algebraic tori are rational.*

Remark: The birational classification of three-dimensional algebraic tori is proved by Kunyavskii [Ku]. See [Ku, Theorem 1] for details. Note that the notation for various integral representations $\rho\colon G \longrightarrow GL_3(\mathbb{Z})$ in [Ku, Theorem 1] is different from that in [Ta]; however, a correspondence of these two systems of notation can be found in [Ku, pp. 9–11].

COROLLARY 2.5: *Let L be a finite Galois field extension of a field K with Galois group G, $\rho\colon G \to GL_3(\mathbb{Z})$ a group homomorphism. Let G act on the rational function field $L(x_1, x_2, x_3)$ as described before. If ρ is a decomposable representation, then $L(x_1, x_2, x_3)^G$ is rational over K.*

Proof: Since ρ is decomposable, we may assume that $\tau(x_1), \tau(x_2) \in < x_1, x_2 >$, $\tau(x_3) = x_3$ or x_3^{-1} for any $\tau \in G$ where $< x_1, x_2 >$ is the multiplicative subgroup generated by x_1 and x_2.

Define $y = (1+x_3)/(1-x_3)$ if $\operatorname{char} K \neq 2$ (resp. $y = 1/(1+x_3)$ if $\operatorname{char} K = 2$). Then $\tau(y) = y$ or $-y$ (resp. $\tau(y) = y$ or $y+1$). Apply Theorem 2.1. It suffices to consider the rationality of $L(x_1, x_2)^G$. But $L(x_1, x_2)^G$ is rational over K by Theorem 2.4. ∎

3. The first step of the proof

We shall prove Theorem 1.4 and Theorem 1.5 together. Thus in this section, K is any field unless otherwise specified, σ: $K(x_1, x_2, x_3) \longrightarrow K(x_1, x_2, x_3)$ is the K-automorphism considered in Theorem 1.4 and Theorem 1.5. Recall the definitions of $\sigma_1, \sigma_2, \sigma_3$ and $f_1(T), f_2(T), f_3(T)$ there.

For $1 \le i \le 3$, define $V_i = Kv_i + Kw_i$ with σ acting on the rational function field of six variables over K, $K(v_1, w_1, v_2, w_2, v_3, w_3)$, by $\sigma \cdot v_i = a_i v_i + b_i w_i$, $\sigma \cdot w_i = c_i v_i + d_i w_i$. We may identify x_i with v_i/w_i.

PROPOSITION 3.1: *If $f_i(T)$ is reducible for some i, then $K(x_1, x_2, x_3)^{<\sigma>}$ is rational over K.*

Proof: We may assume that $f_3(T)$ is reducible. Thus the eigenvalues of σ_3 lie in K and we may find $v_4, w_4 \in V_3$ such that $V_3 = Kv_4 + Kw_4$ and w_4 is an eigenvector of σ when σ is restricted to V_3. Clearly, $K(x_3) = K(y)$ where $y = v_4/w_4$; moreover, $\sigma(y) = \alpha y + \beta$ for some $\alpha, \beta \in K$ with $\alpha \neq 0$. By Theorem 2.2, $K(x_1, x_2, x_3)^{<\sigma>} = K(x_1, x_2, y)^{<\sigma>}$ is rational provided that $K(x_1, x_2)^{<\sigma>}$ is rational. But $K(x_1, x_2)^{<\sigma>}$is rational by Theorem 2.3. ∎

Because of Proposition 3.1 we will assume that $f_1(T), f_2(T), f_3(T) \in K[T]$ are separable irreducible from now on. Note that this assumption will take care of the remaining part of Theorem 1.4 and parts of Theorem 1.5.

Let α_i, β_i be the roots of $f_i(T) = 0$ in some extension field of K. Let $L = K(\alpha_1, \alpha_2, \alpha_3)$ and $G = \mathrm{Gal}(L/K)$ be the Galois group of L over K. Note that $|G| = 2, 4$ or 8. Define

$$\lambda_i = \alpha_i/\beta_i \quad \text{for } 1 \le i \le 3.$$

Note that $\mathrm{ord}(\sigma_i) = \mathrm{ord}(\lambda_i)$. For each $1 \le i \le 3$, find a vector $u_i \in V_i$ such that u_i and $\sigma \cdot u_i$ form a basis of V_i. (This is possible because $f_i(T)$ is irreducible.) Define $Y_i, Z_i \in V_i \otimes_K L$ by

$$Y_i = (\sigma - \beta_i)u_i, \quad Z_i = (\sigma - \alpha_i)u_i.$$

We shall extend the actions of σ and G to $L(x_1, x_2, x_3)$ and $L(v_1, w_1, v_2, w_2, v_3, w_3)$ by requiring $\sigma(\alpha_i) = \alpha_i, \tau(x_i) = x_i, \tau(v_i) = v_i, \tau(w_i) = w_i$ for any $1 \le i \le 3$ and any $\tau \in G$. Define $X_i \in L(v_i, w_i)$ by $X_i = Y_i/Z_i$. It follows that $L(x_i) = L(X_i)$ for $1 \le i \le 3$. It is easy to verify that

$$\sigma \cdot Y_i = \alpha_i Y_i, \quad \sigma \cdot Z_i = \beta_i Z_i, \quad \sigma \cdot X_i = \lambda_i X_i.$$

Moreover, for any $\tau \in G$, we find that

$$\tau \cdot Y_i = \begin{cases} Z_i & \text{if } \tau \cdot \alpha_i = \beta_i, \\ Y_i & \text{if } \tau \cdot \alpha_i = \alpha_i. \end{cases}$$

Hence for any $\tau \in G$,

$$\tau \cdot X_i = \begin{cases} 1/X_i & \text{if } \tau \cdot \alpha_i = \beta_i, \\ X_i & \text{if } \tau \cdot \alpha_i = \alpha_i. \end{cases}$$

Note that

$$\begin{aligned} K(x_1, x_2, x_3)^{<\sigma>} &= \{L(x_1, x_2, x_3)^G\}^{<\sigma>} \\ &= L(x_1, x_2, x_3)^{<G,\sigma>} \\ &= L(X_1, X_2, X_3)^{<G,\sigma>}. \end{aligned}$$

Let $< X_1, X_2, X_3 > := \{X_1^{n_1} X_2^{n_2} X_3^{n_3} \in L(X_1, X_2, X_3) \setminus \{0\} : n_1, n_2, n_3 \in \mathbb{Z}\}$ and define a G-equivariant map φ by

$$\begin{aligned} \varphi\colon\ & < X_1, X_2, X_3 > \longrightarrow L^\times \\ & X_1^{n_1} X_2^{n_2} X_3^{n_3} \mapsto \sigma(X_1^{n_1} X_2^{n_2} X_3^{n_3}) / X_1^{n_1} X_2^{n_2} X_3^{n_3}. \end{aligned}$$

Since $< X_1, X_2, X_3 >$ is isomorphic to a free abelian group of rank three, it follows that Ker φ is a free abelian group of rank $k \leq 3$. Thus $\operatorname{Ker} \varphi = < M_1, \ldots, M_k >$ where each $M_j = X_1^{a_{1j}} X_2^{a_{2j}} X_3^{a_{3j}}$ with $a_{ij} \in \mathbb{Z}$. It follows that $L[X_1, X_2, X_3, 1/(X_1 X_2 X_3)]^{<\sigma>} = L[M_1, \ldots, M_k]$. Since $L(X_1, X_2, X_3)^{<\sigma>}$ equals the quotient field of $L[X_1, X_2, X_3]^{<\sigma>}$ (see the proof of [AHK, Theorem 3.1]), it follows that $K(x_1, x_2, x_3)^{<\sigma>} = L(M_1, \ldots, M_k)^G$.

LEMMA 3.2: *For each $\tau \in G$, $\tau \cdot M_l = M_1^{b_{1,l}} \cdots M_k^{b_{k,l}}$ for $1 \leq l \leq k$ where $b_{i,l} \in \mathbb{Z}$. In particular, $L(M_1, \ldots, M_k)^G$ is the function field of some k-dimensional algebraic torus defined over K and split by L.*

Proof: Since φ is G-equivariant, it follows that Ker φ inherits a G-module structure from that of $< X_1, X_2, X_3 >$. Thus $\tau \cdot M_l \in < M_1, \ldots, M_k >$ for any $\tau \in G$. ∎

PROPOSITION 3.3: *Suppose that $f_i(T) \in K[T]$ is separable irreducible for each $1 \leq i \leq 3$. Assume furthermore that either $|G| = 2$ or $\operatorname{ord}(\sigma_i)$ is infinite for some $1 \leq i \leq 3$. Then $K(x_1, x_2, x_3)^{<\sigma>}$ is rational over K.*

Proof: If $\operatorname{ord}(\sigma_i) = \operatorname{ord}(\lambda_i)$ is infinite, then the rank of the image of φ is ≥ 1 and rank(Ker φ) ≤ 2. If rank(Ker(φ)) $= 2$, then $K(x_1, x_2, x_3)^{<\sigma>} = L(M_1, M_2)^G$ is rational by Theorem 2.4. If $\mathrm{rank}(\mathrm{Ker}\,\varphi) = 1$, then $L(M_1)^G$ is of transcendental degree one and $K(x_1, x_2, x_3)^{<\sigma>}$ is rational by Lüroth's Theorem.

Now assume that $|G| = 2$. Write $G = <\tau>$. By Reiner's Theorem on integral representations of cyclic groups of order p [Sw1, Theorem 4.19, p. 74], there exist $N_1, \ldots, N_k$ such that $< M_1, \ldots, M_k > = < N_1, \ldots, N_k >$ and $\tau \cdot N_i = N_i^{\epsilon_i}$ where $\epsilon_i = 1$ or -1. If $\mathrm{char}\, K \neq 2$, define $A_i = (1-N_i)/(1+N_i)$. We find $\tau \cdot A_i = \epsilon_i A_i$. Thus $L(M_1, \ldots, M_k) = L(N_1, \ldots, N_k) = L(A_1, \ldots, A_k)$. Apply Theorem 2.1 to find $B_1, \ldots, B_k$ such that $\tau(B_i) = B_i$ and $L(A_1, \ldots, A_k) = L(B_1, \ldots, B_k)$. Thus $L(M_1, \ldots, M_k)^G = L(B_1, \ldots, B_k)^G = L^G(B_1, \ldots, B_k) = K(B_1, \ldots, B_k)$ is rational over K. If char $K = 2$, define $A_i = 1/(1 + N_i)$. We find that $\tau \cdot A_i = A_i$ or $A_i + 1$ according to $\epsilon_i = 1$ or -1. Use Theorem 2.1 again and proceed as before. We find $K(x_1, x_2, x_3)^G$ is also rational. ∎

Because of Proposition 3.1 and Proposition 3.3, from now on we shall assume that (i) each $f_i(T) \in K[T]$ is separable irreducible, (ii) $|G| = 4$ or 8, and (iii) ord $(\sigma_i) = \mathrm{ord}(\lambda_i)$ is finite for $1 \leq i \leq 3$. We shall analyze the orders of λ_i in the remaining part of this section. (Remember that $\mathrm{ord}(\lambda_i)$ is an odd integer whenever char $K = 2$.)

Let $o_i = \mathrm{ord}(\lambda_i)$ and $d = \gcd\{o_1, o_2, o_3\}$. If $\{i, j, k\} = \{1, 2, 3\}$, define $e_i = \gcd\{o_j, o_k\}/d$ and $h_i = o_i/(d e_j e_k)$. Since $\gcd\{e_j, e_k\} = 1$, all the e_i and h_i are integers.

In summary, we have the following properties whose proof may be verified by comparing the exponents of a fixed prime number.

(i) $o_1 = de_2e_3h_1$, $o_2 = de_1e_3h_2$, $o_3 = de_1e_2h_3$;

(ii) $\gcd\{o_1, o_2, o_3\} = d$, $\gcd\{o_1, o_2\} = de_3$, $\gcd\{o_1, o_3\} = de_2$, $\gcd\{o_2, o_3\} = de_1$;

(iii) e_1, e_2, e_3 are pairwisely relatively prime; h_1, h_2, h_3 are pairwisely relatively prime; $\gcd\{e_i, h_i\} = 1$ for $1 \leq i \leq 3$;

(iv) $\mathrm{lcm}\{o_1, o_2, o_3\} = de_1e_2e_3h_1h_2h_3$.

Define $E = de_1e_2e_3$ and $F = Eh_1h_2h_3$. Then $\sigma^E \cdot X_i = \lambda_i^E X_i$ for $1 \leq i \leq 3$ and $\mathrm{ord}(\sigma) = F$.

Define $y_i = X_i^{h_i}$ for $1 \leq i \leq 3$. It follows that $L(X_1, X_2, X_3)^{<\sigma^E>} = L(y_1, y_2, y_3)$. Moreover, if $\tau \in G$, then

$$\tau \cdot y_i = \begin{cases} y_i & \text{if } \tau \cdot \alpha_i = \alpha_i, \\ 1/y_i & \text{if } \tau \cdot \alpha_i = \beta_i. \end{cases}$$

We write $\zeta_i = \lambda_i^{h_i}$. Thus $\sigma \cdot y_i = \zeta_i y_i$ and $\mathrm{ord}(\zeta_1) = de_2e_3$, $\mathrm{ord}(\zeta_2) = de_1e_3$, $\mathrm{ord}(\zeta_3) = de_1e_2$. Note that the order of $\sigma \in \mathrm{Aut}(L(y_1, y_2, y_3))$ is $E(= de_1e_2e_3)$.

Now consider the action of G on $L(y_1, y_2, y_3)$.

Note that $[K(\alpha_i) : K] = 2$ for $1 \leq i \leq 3$. If $|G| = 4$, then either of $K(\alpha_1), K(\alpha_2)$, $K(\alpha_3)$ are distinct subfields of L or precisely two among $K(\alpha_1), K(\alpha_2), K(\alpha_3)$ will coincide; in the latter situation, we may assume that $K(\alpha_1) = K(\alpha_2)$ without loss of generality. Hence we have the following possibilities.

CASE 1: $|G| = 4$ and $G =< \tau_1, \tau_2 >$.

CASE 1.1: $K(\alpha_1) = K(\alpha_2)$.

$$\tau_1 \colon \alpha_1 \mapsto \beta_1, \quad \alpha_2 \mapsto \beta_2, \quad \alpha_3 \mapsto \alpha_3,$$
$$\tau_2 \colon \alpha_1 \mapsto \alpha_1, \quad \alpha_2 \mapsto \alpha_2, \quad \alpha_3 \mapsto \beta_3.$$

CASE 1.2: All the $K(\alpha_i)$ are distinct.

$$\tau_1 \colon \alpha_1 \mapsto \beta_1, \quad \alpha_2 \mapsto \alpha_2, \quad \alpha_3 \mapsto \beta_3,$$
$$\tau_2 \colon \alpha_1 \mapsto \alpha_1, \quad \alpha_2 \mapsto \beta_2, \quad \alpha_3 \mapsto \beta_3.$$

CASE 2: $|G| = 8$ and $G =< \tau_1, \tau_2, \tau_3 >$.

$$\tau_1 \colon \alpha_1 \mapsto \beta_1, \quad \alpha_2 \mapsto \alpha_2, \quad \alpha_3 \mapsto \alpha_3,$$
$$\tau_2 \colon \alpha_1 \mapsto \alpha_1, \quad \alpha_2 \mapsto \beta_2, \quad \alpha_3 \mapsto \alpha_3,$$
$$\tau_3 \colon \alpha_1 \mapsto \alpha_1, \quad \alpha_2 \mapsto \alpha_2, \quad \alpha_3 \mapsto \beta_3.$$

Now we will find new variables w, z_2, z_3 such that $L(y_1, y_2, y_3) = L(w, z_2, z_3)$ satisfying $\sigma \cdot w = \zeta \cdot w, \sigma \cdot z_2 = z_2, \sigma \cdot z_3 = z_3$ where ζ is a primitive E-th root of unity.

In fact, we will choose ζ as follows. Let ζ be a primitive E-th root of unity such that $\zeta^{e_1} = \zeta_1$. Note that ζ is a generator of the cyclic group $< \zeta_1, \zeta_2, \zeta_3 >$. For $2 \leq i \leq 3$, since $< \zeta_i >$ and $< \zeta^{e_i} >$ are the same cyclic subgroup, we may write $\zeta_i = \zeta^{c_i e_i}$ for an integer c_i satisfying the property that $\gcd\{c_i, E/e_i\} = 1$.

Since e_1 and $c_2 e_2$ are relatively prime, we may find integers a and b so that $ae_1 + bc_2 e_2 = 1$.

Define $w = y_1^a y_2^b$, $z_2 = y_1^{-c_2 e_2} y_2^{e_1}$, $z_3 = y_1^{-ac_3 e_3} y_2^{-bc_3 e_3} y_3$. Since

$$\det \begin{pmatrix} a & -c_2 e_2 & -ac_3 e_3 \\ b & e_1 & -bc_3 e_3 \\ 0 & 0 & 1 \end{pmatrix} = 1,$$

we find that $L(y_1, y_2, y_3) = L(w, z_2, z_3)$. It is easy to see that $\sigma \cdot w = \zeta \cdot w, \sigma \cdot z_2 = z_2, \sigma \cdot z_3 = z_3$. It follows that $L(y_1, y_2, y_3)^{<\sigma>} = L(z_1, z_2, z_3)$ where $z_1 = w^E$.

Note that, for $1 \le i \le 3$, y_i^{E/e_i} is fixed by σ. Thus it belongs to $L(z_1, z_2, z_3)$. Although we will not use the following explicit formula, we record it here: $y_1^{E/e_1} = z_1 z_2^{-bE/e_1}$, $y_2^{E/e_2} = z_1^{c_2} z_2^{aE/e_2}$, $y_3^{E/e_3} = z_1^{c_3} z_3^{E/e_3}$.

It is routine to verify the actions of G on z_1, z_2, z_3. We list them as follows.

CASE 1.1: $K(\alpha_1) = K(\alpha_2)$.

$$\tau_1\colon z_1 \mapsto z_1^{-1}, \quad z_2 \mapsto z_2^{-1}, \quad z_3 \mapsto z_3 w^{2c_3e_3},$$
$$\tau_2\colon z_1 \mapsto z_1, \quad z_2 \mapsto z_2, \quad z_3 \mapsto z_3^{-1} w^{-2c_3e_3}.$$

CASE 1.2: All the $K(\alpha_i)$ are distinct.

$$\tau_1\colon z_1 \mapsto y_1^{-aE} y_2^{bE}, \quad z_2 \mapsto z_2^{-1} y_2^{2e_1}, \quad z_3 \mapsto z_3^{-1} y_2^{-2bc_3e_3},$$
$$\tau_2\colon z_1 \mapsto y_1^{aE} y_2^{-bE}, \quad z_2 \mapsto z_2 y_2^{-2e_1}, \quad z_3 \mapsto z_3^{-1} y_1^{-2ac_3e_3}.$$

CASE 2: $|G| = 8$.

$$\tau_1\colon z_1 \mapsto y_1^{-aE} y_2^{bE}, \quad z_2 \mapsto z_2^{-1} y_2^{2e_1}, \quad z_3 \mapsto z_3 y_1^{2ac_3e_3},$$
$$\tau_2\colon z_1 \mapsto y_1^{aE} y_2^{-bE}, \quad z_2 \mapsto z_2 y_2^{-2e_1}, \quad z_3 \mapsto z_3 y_2^{2bc_3e_3},$$
$$\tau_3\colon z_1 \mapsto z_1, \quad z_2 \mapsto z_2, \quad z_3 \mapsto z_3 y_3^{-2}.$$

In view of Lemma 3.2 with $\{M_1, \ldots, M_k\} = \{z_1, z_2, z_3\}$, it is necessary that (i) $w^{2c_3e_3} \in < z_1, z_2, z_3 >$ for Case 1.1, (ii) $y_1^{2ac_3e_3}, y_2^{2e_1}, y_2^{2bc_3e_3} \in < z_1, z_2, z_3 >$ for Case 1.2, (iii) $y_1^{2ac_3e_3}, y_2^{2e_1}, y_2^{2bc_3e_3}, y_3^2 \in < z_1, z_2, z_3 >$ for Case 2.

LEMMA 3.4:

(i) *In Case 1.1, it is necessary that* $E/e_3 = 1$ *or* 2.

(ii) *In both Case 1.2 and Case 2, it is necessary that* $E = 1$ *or* 2.

Proof: (i) Since $w^{2c_3e_3} \in < z_1, z_2, z_3 >$, $w^{2c_3e_3}$ is fixed by σ. On the other hand, $\sigma \cdot w = \zeta \cdot w$. Thus $\zeta^{2c_3e_3} = 1$. Since $\zeta^{c_3e_3} = \zeta_3$, it follows that $\zeta_3^2 = 1$. Hence 2 is divisible by E/e_3.

(ii) Since $\sigma \cdot y_1 = \zeta_1 \cdot y_1, \sigma \cdot y_2 = \zeta_2 \cdot y_2$, it follows that $2ac_3e_3$ is divisible by E/e_1, and both $2e_1$ and $2bc_3e_3$ are divisible by E/e_2. It follows that all of $2ac_3e_1e_3, 2e_1e_2, 2bc_3e_2e_3$ are divisible by E. Since $ac_3e_1e_3, e_1e_2, bc_3e_2e_3$ are relatively prime, hence E will divide 2. ∎

4. Main results

We will retain the assumptions and notation in Section 3 until we finish the proof of Proposition 4.3. In particular, recall $K(x_1,x_2,x_3)^{<\sigma>} = L(z_1,z_2,z_3)^G$.

LEMMA 4.1: *In both Case 1 and Case 2, if $E = 1$, then $K(x_1,x_2,x_3)^{<\sigma>}$ is rational over K.*

Proof: Note that $K(x_1,x_2,x_3)^{<\sigma>} = L(y_1,y_2,y_3)^{<\sigma,G>}$. Because $E = 1$, all the y_i are fixed by σ. It remains to consider $L(y_1,y_2,y_3)^G$, which is rational over K since the associated representation $G \to GL_3(\mathbb{Z})$ is decomposable and thus we may apply Corollary 2.5. ∎

LEMMA 4.2: *In both Case 1.2 and Case 2, if $E = 2$ and $d = 1$, then $K(x_1,x_2,x_3)^{<\sigma>}$ is rational over K.*

Proof: We will prove the situation $e_3 = 2$ only; the proof of other situations are almost the same.

Note that $\sigma\cdot y_1 = -y_1, \sigma\cdot y_2 = -y_2, \sigma\cdot y_3 = y_3$. Define $w_1 = y_1y_2, w_2 = y_1^{-1}y_2$. It follows that $L(y_1,y_2,y_3)^{<\sigma>} = L(w_1,w_2,y_3)$. The actions of G on w_1,w_2,y_3 for Case 1.2 (resp. Case 2) arise from a decomposable integral representation. Thus $L(w_1,w_2,y_3)^G$ is rational over K by Corollary 2.5. ∎

PROPOSITION 4.3: *Assume that $f_i(T) \in K[T]$ is separable irreducible and $\operatorname{ord}(\lambda_i) < \infty$ for $1 \le i \le 3$. Then $K(x_1,x_2,x_3)^{<\sigma>}$ is not rational over K if and only if $|G| = 8$ and $\operatorname{ord}(\lambda_i)$ is an even integer for $1 \le i \le 3$. If $K(x_1,x_2,x_3)^{<\sigma>}$ is not rational over K, it is not stably rational over K; in this situation, $K(x_1,x_2,x_3)^{<\sigma>}$ is isomorphic over K to the function field of the algebraic torus W_5 (see [Ta, p. 187]) defined over K.*

Proof: The case when $|G| = 2$ is solved by Proposition 3.3. Hence it suffices to consider the case $|G| = 4$ or 8. We will adopt the notation Case 1.1, 1.2, and Case 2 in Section 3.

CASE 1.1: $K(\alpha_1) = K(\alpha_2)$.

By Lemma 3.4 (i), $E = e_3$ or $2e_3$. Thus we may write $w^{2c_3e_3} = (w^E)^{2c_3e_3/E} = z_1^{\varepsilon c_3}$ where $\varepsilon = 2$ if $E = e_3$, and $\varepsilon = 1$ if $E = 2e_3$. The actions of G on z_1,z_2,z_3 are given by

$$\tau_1\colon z_1 \mapsto z_1^{-1}, \quad z_2 \mapsto z_2^{-1}, \quad z_3 \mapsto z_1^{\varepsilon c_3}z_3,$$
$$\tau_2\colon z_1 \mapsto z_1, \quad z_2 \mapsto z_2, \quad z_3 \mapsto z_1^{-\varepsilon c_3}z_3^{-1}.$$

By Corollary 2.5, $L(z_1,z_2,z_3)^G$ is rational over K.

CASE 1.2: All the $K(\alpha_i)$ are distinct.

By Lemma 3.4 (ii) either $E = 1$ or $E = 2$.

If $E = 1$ (resp. $E = 2$ and $d = 1$), we may apply Lemma 4.1 (resp. Lemma 4.2) to conclude that $K(x_1, x_2, x_3)^{<\sigma>}$ is rational. It remains to consider the situation when $E = d = 2$.

Note that $\sigma \cdot y_i = -y_i$ for $1 \leq i \leq 3$. Define $z_1 = y_2y_3, z_2 = y_1y_3, z_3 = y_1y_3^{-1}$. Since $[L(y_1, y_2, y_3) : L(z_1, z_2, z_3)] = 2$, it follows that $L(y_1, y_2, y_3)^{<\sigma>} = L(z_1, z_2, z_3)$. The actions of G on z_1, z_2, z_3 are given by

$$\begin{aligned} &\tau_1 \colon z_1 \mapsto z_1 z_2^{-1} z_3, \quad z_2 \mapsto z_2^{-1}, \quad z_3 \mapsto z_3^{-1}, \\ &\tau_2 \colon z_1 \mapsto z_1^{-1}, \quad z_2 \mapsto z_3, \quad z_3 \mapsto z_2. \end{aligned}$$

The matrices associated to τ_2 and $\tau_1\tau_2$ are

$$\tau_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \tau_1\tau_2 = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix},$$

which is the Case W_{12} in [Ta, p. 174]. The corresponding 3-dimensional algebraic torus is rational by [Ku, Theorem 1]. See [Ku, p. 9, line 7] also.

CASE 2: $| G |= 8$.

By Lemma 3.4 (ii) either $E = 1$ or $E = 2$. As in Case 1.2 it suffices to consider the situation when $E = d = 2$.

Again $\sigma \cdot y_i = -y_i$ for $1 \leq i \leq 3$. Thus $L(y_1, y_2, y_3)^{<\sigma>} = L(z_1, z_2, z_3)$ where $z_1 = y_2y_3, z_2 = y_1y_3, z_3 = y_1y_3^{-1}$. The actions of G on z_1, z_2, z_3 are given by

$$\begin{aligned} &\tau_1 \colon z_1 \mapsto z_1, \quad z_2 \mapsto z_3^{-1}, \quad z_3 \mapsto z_2^{-1}, \\ &\tau_2 \colon z_1 \mapsto z_1^{-1} z_2 z_3^{-1}, \quad z_2 \mapsto z_2, \quad z_3 \mapsto z_3, \\ &\tau_3 \colon z_1 \mapsto z_1 z_2^{-1} z_3, \quad z_2 \mapsto z_3, \quad z_3 \mapsto z_2. \end{aligned}$$

Thus the matrices associated to $\tau_2\tau_3, \tau_1\tau_2$ and $\tau_1\tau_2\tau_3$ are

$$\tau_2\tau_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \tau_1\tau_2 = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}, \quad \tau_1\tau_2\tau_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

which is the case W_5 in [Ta, p. 187]. The corresponding 3-dimensional algebraic torus is not stably rational by [Ku, Theorem 1]. See [Ku, p. 9, line 5 from the bottom] also.

In conclusion, the only non-rational situation happens if and only if $| G |= 8$ and $d = 2$. ∎

Proof of Theorem 1.4: Applying Proposition 3.1, Proposition 3.3 and Proposition 4.3, we are able to obtain the rationality criterion we desire together with the property that, if $K(x_1, x_2, x_3)^{<\sigma>}$ is not rational, then it is not stably rational.

It remains to prove that, if the fixed field is not rational, then it is not retract rational. By Proposition 4.3 and Theorem 1.3, we find that the algebraic torus W_5 [Ta, p. 187] defined over K is not retract rational over K. It follows again from Proposition 4.3 that any non-rational fixed field is not retract rational. ∎

Proof of Theorem 1.5: If some $f_i(T) \in K[T]$ is reducible, apply Proposition 3.1.

Hence we may assume that $f_i(T) \in K[T]$ is irreducible for $1 \le i \le 3$ from now on.

CASE 1: If all $f_i(T)$ are separable, apply Proposition 4.3. Note that none of $\mathrm{ord}(\lambda_i)$ is an even integer.

CASE 2: Only one of $f_i(T)$ is separable.

We may assume that $f_1(T)$ is separable and $f_2(T) = T^2 + a$, $f_3(T) = T^2 + b$. We may assume that $\sigma(x_2) = a/x_2, \sigma(x_3) = b/x_3$. Clearly $\sigma^2(x_2) = x_2$ and $\sigma^2(x_3) = x_3$.

Let $L = K(\alpha)$ where $f_1(T) = (T - \alpha)(T - \beta)$. Define $\lambda = \alpha/\beta$ and $\tau\colon \alpha \mapsto \beta$; $G = <\tau>$ is the Galois group of L over K.

Apply the standard arguments in this paper. Write $L(x_1) = L(X)$ with $\sigma \cdot X = \lambda X, \tau \cdot X = 1/X$.

CASE 2.1: $\mathrm{ord}(\lambda) = \infty$.

Then $K(x_1, x_2, x_3)^{<\sigma>} = L(X_1, x_2, x_3)^{<G,\sigma>} = \{L(X, x_2, x_3)^{<\sigma^2>}\}^{<\sigma,G>} = L(x_2, x_3)^{<\sigma,G>} = \{L(x_2, x_3)^G\}^{<\sigma>} = K(x_2, x_3)^{<\sigma>}$ is rational by Theorem 1.1.

CASE 2.2: $\mathrm{ord}(\lambda) < \infty$.

It is necessary that $\mathrm{ord}(\lambda)$ is odd. Call $m = \mathrm{ord}(\lambda)$. Then $\mathrm{ord}(\sigma) = 2m$.

By Lüroth's Theorem, $K(x_1)^{<\sigma>} = K(w)$ for some $w \in K(x_1)$. By Theorem 1.2, $K(x_2, x_3)^{<\sigma>} = K(u, v)$. Since

$$[K(x_1) : K(w)] = m \quad \text{and} \quad [K(x_2, x_3) : K(u, v)] = 2,$$

it follows that

$$[K(x_1, x_2, x_3) : K(u, v, w)] = 2m = [K(x_1, x_2, x_3) : K(x_1, x_2, x_3)^{<\sigma>}].$$

Thus $K(x_1,x_2,x_3)^{<\sigma>} = K(u,v,w)$ is rational.

CASE 3: Precisely two of $f_i(T)$ are separable.

We may assume that both $f_1(T)$ and $f_2(T)$ are separable and $f_3(T) = T^2 + a$.

Let $L = K(\alpha_1, \alpha_2)$ where $f_i(T) = (T - \alpha_i)(T - \beta_i)$ for $i = 1$ or 2. Let G be the Galois group of L over K. Note that $|G| = 2$ or 4.

As in Case 2, we may assume that $\sigma(x_3) = a/x_3$ and there exists $X_i \in L(x_i)$ for $i = 1$ or 2 such that $K(x_1,x_2,x_3)^{<\sigma>} = L(X_1,X_2,x_3)^{<G,\sigma>}, \sigma(X_i) = \lambda_i X_i$ where $\lambda_i = \alpha_i/\beta_i$.

CASE 3.1: Both λ_1 and λ_2 are roots of unity.

The order of the restriction of σ to $K(x_1,x_2)$ is $\operatorname{lcm}\{\operatorname{ord}(\lambda_1), \operatorname{ord}(\lambda_2)\}$. Call it m. Note that m is an odd integer and $\operatorname{ord}(\sigma) = 2m$.

By Theorem 2.3, $K(x_1,x_2)^{<\sigma>}$ is rational. Now proceed as in Case 2.2. It is easy to see that $K(x_1,x_2,x_3)^{<\sigma>}$ is rational.

CASE 3.2: The multiplicative group $< \lambda_1, \lambda_2 >$ is isomorphic to $\mathbb{Z}^2$.

Then $K(x_1,x_2,x_3)^{<\sigma>} = L(X_1,X_2,x_3)^{<G,\sigma>} = \{L(X_1,X_2,x_3)^{<\sigma^2>}\}^{<\sigma,G>} = L(x_3)^{<G,\sigma>} = K(x_3)^{<\sigma>}$ is rational.

CASE 3.3: The multiplicative group $< \lambda_1, \lambda_2 > \simeq < \mu > \times < \nu >$ where $< \mu >$ is a finite group of order m and $< \nu > \simeq \mathbb{Z}$. Clearly m is an odd integer.

It is not difficult to find $Y_1, Y_2 \in < X_1, X_2 >$ such that $< X_1, X_2 > = < Y_1, Y_2 >$ and $\sigma(Y_1) = \mu Y_1, \sigma(Y_2) = \nu Y_2$.

Now $K(x_1,x_2,x_3)^{<\sigma>} = L(Y_1,Y_2,x_3)^{<G,\sigma>} = \{L(Y_1,Y_2,x_3)^{<\sigma^2>}\}^{<\sigma,G>} = L(Y_1^m,x_3)^{<G,\sigma>}$. Define $y = 1/(1+Y_1^m)$. Then $\sigma \cdot y = y$ and $\tau \cdot y = y$ or $y+1$ for any $\tau \in G$, because G sends the multiplicative group $< Y_1^m >$ onto itself by Lemma 3.2. Apply Theorem 2.1 and find $z \in L(y,x_3)$ with $\sigma \cdot z = \tau \cdot z = z$ for any $\tau \in G$. Thus $L(y,x_3)^{<\sigma,G>} = L(x_3)^{<\sigma,G>}(z) = K(x_3)^{<\sigma>}(z)$ is rational. ∎

Finally, we reformulate Theorem 1.3. as follows.

THEOREM 4.4: *Let K be any field with* $\operatorname{char} K \neq 2$ *and* $n \geq 3$. *If*

$$\sigma: K(x_1,\ldots,x_n) \longrightarrow K(x_1,\ldots,x_n)$$

is a K-automorphism defined by $\sigma \cdot x_i = a_i/x_i$ where $a_i \in K \setminus \{0\}$ for $1 \leq i \leq n$, then $K(x_1,\ldots,x_n)^{<\sigma>}$ is not rational over K if and only if

$$[K(\sqrt{a_1},\ldots,\sqrt{a_n}) : K] \geq 8.$$

If $K(x_1, \ldots, x_n)^{<\sigma>}$ is not rational, it is not retract rational.

Proof: Let $< a_1, \ldots, a_n >$ generate a group of order 2^k in $K^\times/K^{\times 2}$. It follows that $2^k = [K(\sqrt{a_1}, \ldots, \sqrt{a_n}) : K]$.

If $k \leq 2$, we shall prove that $K(x_1, \ldots, x_n)^{<\sigma>}$ is rational. Suppose that $k = 2$ and a_1, a_2 generate $< a_1, \ldots, a_n >$ in $K^\times/K^{\times 2}$. If $a_3 = a_1 a_2 b_3^2$ for some $b_3 \in K \setminus \{0\}$, define $y_3 = x_3/(b_3 x_1 x_2)$; if $a_4 = a_1 b_4^2$, define $y_4 = x_4/(b_4 x_1)$. In this way we find $y_3, \ldots, y_n$ such that $K(x_1, \ldots, x_n) = K(x_1, x_2, y_3, \ldots, y_n)$ and $\sigma \cdot y_i = 1/y_i$ for $3 \leq i \leq n$. Define $z_i = (1 + y_i)/(1 - y_i)$ for $3 \leq i \leq n$. We find $\sigma \cdot z_i = -z_i$. Apply Theorem 2.2 and Theorem 1.1. We find that $K(x_1, x_2, z_3, \ldots, z_n)^{<\sigma>}$ is rational. The case when $k = 1$ or 0 can be treated similarly.

Assume $k \geq 3$. Without loss of generality we may assume that $a_1, a_2, \ldots, a_k$ generate $< a_1, \ldots, a_n >$ in $K^\times/K^{\times 2}$. Proceed as above and find $z_{k+1}, \ldots, z_n$ such that $K(x_1, \ldots, x_n) = K(x_1, \ldots, x_k, z_{k+1}, \ldots, z_n)$ and $\sigma \cdot z_i = -z_i$ for $k + 1 \leq i \leq n$. By Theorem 2.1 find $u_{k+1}, \ldots, u_n$ such that $K(x_1, \ldots, x_n) = K(x_1, \ldots, x_k, u_{k+1}, \ldots, u_n)$ with $\sigma \cdot u_i = u_i$ for $k+1 \leq i \leq n$. It suffices to show that $K(x_1, \ldots, x_k)^{<\sigma>}$ is not retract rational.

Let $L = K(\sqrt{a_4}, \ldots, \sqrt{a_k})$ and extend the action of σ to $L(x_1, \ldots, x_k)$ by $\sigma \cdot \sqrt{a_j} = \sqrt{a_j}$ for $4 \leq j \leq k$. If $L(x_1, \ldots, x_k)^{<\sigma>}$ is not retract rational over L, then $K(x_1, \ldots, x_k)^{<\sigma>}$ is not retract rational.

Within $L(x_1, \ldots, x_k)$, define $y_j = x_j/\sqrt{a_j}$ for $4 \leq j \leq k$. Then $\sigma \cdot y_j = 1/y_j$. Thus we may find $u_4, \ldots, u_k$ such that $L(x_1, \ldots, x_k) = L(x_1, x_2, x_3, u_4, \ldots, u_k)$ with $\sigma \cdot u_j = u_j$ for $4 \leq j \leq k$. Since $[L(\sqrt{a_1}, \sqrt{a_2}, \sqrt{a_3}) : L] = 8$, $L(x_1, x_2, x_3)^{<\sigma>}$ is not retract rational by Theorem 1.3. It follows that $L(x_1, x_2, x_3, u_4, \ldots, u_k)^{<\sigma>}$ is not retract rational over L. ∎

Remark: We don't know how to generalize Theorem 4.4, i.e. what would be a necessary and sufficient condition for $K(x_1, \ldots, x_n)^{<\sigma>}$ being rational over K if $n \geq 4$, $\operatorname{char} K \neq 2$ and $\sigma \cdot x_i = (a_i x_i + b_i)/(c_i x_i + d_i)$ where $a_i d_i - b_i c_i \neq 0$? Using Theorem 1.4 it is not difficult to check that the following three assumptions guarantee that the fixed field is not retract rational: (i) for $1 \leq i \leq n$, each $f_i(T)$ is separable irreducible, (ii) the Galois group of $f_1(T) f_2(T) \cdots f_n(T)$ is of order 2^n, and (iii) at least three among $\operatorname{ord}(\sigma_i)$ are even integers where $1 \leq i \leq n$.

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